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GENERALIZED 3-MANIFOLDS WHOSE NONMANIFOLD SET HAS NEIGHBORHOODS BOUNDED BY TORI

BY

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ABSTRACT. We show that all compact, ANR, generalized 3-manifolds whose nonmanifold set is 0-dimensional and has a neighborhood system bounded by tori are cell-like images of compact 3-manifolds if and only if the Poincaré conjecture is true. We also discuss to what extent the assumption of the Poincaré conjecture can be replaced by other hypotheses.

1. Introduction. In the search for a characterization of topological manifolds, the following arises.

Question 1. *Is every ANR, generalized n-manifold (n finite) the cell-like image of a topological n-manifold?*

Many results exist for $n \geqslant 5$. In this paper we give a partial result for $n = 3$. Our result also gives a sufficient condition for an open 3-manifold to embed in a compact 3-manifold.

Since we restrict our discussion to ANR's (Absolute Neighborhood Retracts), we incorporate this into the following definition. A *generalized n-manifold M* is an n-dimensional retract of an open set of some finite dimensional Euclidean space E^k, with the following local homology property at every point x in M:

$$H_*(M, M - \{x\}) \cong H_*(E^n, E^n - \{0\}).$$

The set of generalized n-manifolds properly contains the set of topological n-manifolds. Generalized manifolds arise as factors and cell-like images of topological manifolds. It is not known if all generalized manifolds are such.

A continuum X in an ANR is *cell-like* if it contracts in any neighborhood of itself. A map from an ANR onto a separable metric space is *cell-like* if the inverse image of every point is cell-like.

Most recent results concerning Question 1 depend on the size of the *nonmanifold set*, the set of points each with no neighborhood homeomorphic to E^n. In **[7]** it is proven that every generalized n-manifold, $n \geqslant 5$, whose nonmanifold set has dimension less than $(n - 1)/2$ is the cell-like image of an n-manifold. A complete affirmative answer to Question 1 for $n \geqslant 5$ has been announced in **[18]**.

Recent survey articles giving more information about the above topics (in various combinations) are **[6]**, **[13]** and **[19]**. For more information about ANR's see **[4]**.

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McMillan has observed that for compact spaces, if $n = 3$ and the nonmanifold set is 0-dimensional, then Question 1 is equivalent to:

Question 2. *If X is a compact generalized* 3-*manifold with* 0-*dimensional nonmanifold set, can the complement M of the nonmanifold set be embedded in a compact* 3-*manifold $\overline{M}$*?

Not every open n-manifold embeds in a compact n-manifold. In fact, there are open n-manifolds that are algebraically very simple (e.g., contractible) that embed in no compact n-manifold. In [**9**, §5] it is shown that the contractible 3-manifolds of [**10**] and [**14**] embed in no compact 3-manifold. Higher dimensional examples are found in [**20**].

To see the equivalence of Questions 1 and 2, we note that if such an embedding does exist, then a different $\overline{M}$ and embedding can be found so that the complement of M in $\overline{M}$ is an intersection of handlebodies [**5**]. We can map $\overline{M}$ to X using the identity on M and mapping each component of $\overline{M} - M$ to the appropriate point of $X - M$. By [**16**, Theorem 3], each component of $\overline{M} - M$ will be a 1-UV intersection of handlebodies and hence cell-like. (See below for a definition of 1-UV.)

If X is a cell-like image under a map f of a compact 3-manifold $\overline{M}'$, then for each x in the nonmanifold set of X, there is an open neighborhood U of $f^{-1}(x)$ such that $U - f^{-1}(x)$ embeds in E^3 [**11**]. By [**15**] the restriction of f to $U - f^{-1}(x)$ is cellular and by [**1**] is approximately a homeomorphism. Thus, a closed subset of M together with a neighborhood of the inverse image of the nonmanifold set can be pieced together to get the desired $\overline{M}$. Some of these arguments require orientability near the nonmanifold set. This we have by the local homology properties of generalized manifolds.

Using the above observation we are able to prove the following.

THEOREM 1. *Let $\mathfrak{M}$ be the set of compact generalized* 3-*manifolds satisfying*

(i) *the nonmanifold set is* 0-*dimensional, and*

(ii) *every open neighborhood of the nonmanifold set contains a closed neighborhood of the nonmanifold set each component of which has connected torus boundary.*

Then every M in $\mathfrak{M}$ is the cell-like image of a compact 3-*manifold if and only if the Poincaré conjecture is true.*

Since property (ii) pervades this paper, we give it a name, saying that a space with this property *has neighborhoods bounded by tori*. A topological manifold has neighborhoods bounded by spheres. A result corresponding to Theorem 1 for spaces with neighborhoods bounded by spheres is found in [**8**]. In this case the conclusion is that every M in $\mathfrak{M}$ is a manifold if and only if the Poincaré conjecture is true.

Theorem 1 follows directly from Theorem 2, which is stated and proved in the next section. Theorem 2 does not use the full hypothesis of Theorem 1. It is not known if this is a strict weakening of the hypotheses. Theorem 2 is stated as a criterion for embedding an open 3-manifold in a compact 3-manifold (compare with [**23**]).

The fact that, when $n = 3$, the Poincaré conjecture gets in the way of an affirmative answer to Question 1 is well known. The one point compactification of the locally finite connected sum of a countably infinite number of fake spheres is a generalized 3-manifold, but the manifold set (the complement of the nonmanifold set) embeds in no compact 3-manifold. In this example, the manifold set is not irreducible. In §3 we construct an example showing that the Poincaré conjecture is not avoided by assuming that the manifold set is irreducible.

Assumptions stronger than irreducibility do bear fruit. McMillan has shown, by methods different from ours, that if the manifold set M is the ascending union of knot spaces (complements in E^3 of solid tori), then M in fact embeds in S^3. Our methods reach the same conclusion under the stronger assumption that M is an ascending union of solid tori. This last is stated as an addendum to Theorem 2.

The example referred to above and the example in §3 are both one-point compactifications of open 3-manifolds. In general, such objects are not very friendly. We point out that if M is the one-point compactification of the connected sum of a countably infinite number of homology spheres, then $H_1(M)$ is uncountably generated.

We will finish this section by setting down some definitions and conventions. In this paper, all manifolds and maps of manifolds will be in the PL category. An *open* manifold is noncompact and without boundary; a *closed* manifold is compact and without boundary. A 3-manifold is *prime* if every separating 2-sphere bounds a 3-cell; it is *irreducible* if every 2-sphere bounds a 3-cell. We will use the symbols I^n, S^n and E^n to represent the n-dimensional ball, sphere and Euclidean space respectively. A *torus* (*annulus*, *solid torus*) is a space homeomorphic to $S^1 \times S^1$ ($I \times S^1$, $I^2 \times S^1$). We will use Bd, Cl and Int to denote boundary, closure and interior.

In a solid torus, $I^2 \times S^1$, a curve embedded in the boundary in the same way as Bd $I^2 \times \{p\}$ will be called a *meridian*, and a disk embedded in the solid torus in the same way as $I^2 \times \{p\}$ will be called a *meridional disk*. If M is an n-manifold with boundary, we say that we are *adding a k-handle* to M by attaching the n-cell $I^{n-k} \times I^k$ to Bd M by a homeomorphism defined on $I^{n-k} \times \text{Bd}\, I^k$. A k-cell intersecting a k-handle is *consistent with the product structure* if it intersects in the set $\{p\} \times I^k$ for some p in Int I^{n-k}.

We use Link(J, K) to denote the linking number of two disjoint loops J and K. See [**2**, pp. 480–482] for a geometric description of linking over Z_2. The discussion there works perfectly well over Z.

For basic facts about incompressible surfaces, see [**22**, §1.1]. Generally, the operation of *compression*, removing an essential annulus from the interior of a surface and sewing on a pair of disks to the newly created boundary components, is done on a surface embedded in a 3-manifold. In this paper, the term will also refer to performing the same operation on the domain of a singular map from a 2-manifold into a 3-manifold. Throughout this paper, the word surface should be taken to mean a singular map from a 2-manifold unless it is specified as nonsingular.

Theorem 2 refers to the Freudenthal compactification (also called the Ideal compactification). This can be defined as follows. Let X be σ-compact and let $K_1 \subset K_2 \subset K_3 \subset \ldots$ be a sequence of compact sets so that $X = \bigcup K_i$. An *end* of

X is a sequence $E = \{U_1 \supset U_2 \supset \dots\}$ of nonempty, connected complementary domains of the sets K_i. The *Freudenthal compactification* of X is X with all "points" E as above, a neighborhood base for E being $\{U_1, U_2, \dots\}$. Note that if M is a compact manifold and C is a closed, nonseparating, 0-dimensional set in M, then the Freudenthal compactification of $M - C$ is M.

We list some local properties. A space is *locally contractible at p* if every open set U containing p contains an open set V containing p so that V contracts in U. A space is *k-LC at p* if every open U about p contains an open V about p so that every map of S^k into V extends to a map of I^{k+1} into U. A set X in a space Y is *k-UV* if every open U containing X contains an open V containing X so that every map of S^k into V extends to a map of I^{k+1} into U. An open manifold M will be said to be *1-acyclic at infinity* if every compact set K in M is contained in a compact set K' so that every 1-cycle in $M - K'$ bounds a 2-chain in $M - K$. Lastly, we will say that a set X is *simply connected* mod $A \subset X$ if every loop in X is homotopic to a product of conjugates in X of loops in A.

2. An embedding criterion.

THEOREM 2. *Let $\mathfrak{M}$ be the class of all connected, open 3-manifolds M without boundary satisfying*

(i) *$M = \bigcup^{\infty} K_i$ where each K_i is a connected, compact 3-manifold and $K_i \subset$* Int K_{i+1}.

(ii) Bd K_i *is a union of tori, for all i.*

(iii) *Each component of* Bd K_i *separates M.*

(iv) *M is 1-acyclic at infinity and*

(v) *$\hat{M}$, the Freudenthal compactification of M, is 1-LC.*

Then every M in $\mathfrak{M}$ embeds in a compact 3-manifold if and only if the Poincaré conjecture is true.

ADDENDUM 2.1. *If, in Theorem 2, hypothesis* (ii) *is replaced by* (ii′) *$K_i \cong I^2 \times S^1$, then every M in $\mathfrak{M}$ embeds in S^3 and the assumption of the Poincaré conjecture is unnecessary.*

The two statements above will be proven together. The next lemma shows that property (iv) arises fairly naturally. The proof that follows, due to McMillan, is shorter than our original proof.

LEMMA 2.2. *If M is a connected open 3-manifold and $H_1(M) = 0$, then M is 1-acyclic at infinity.*

PROOF. It suffices to show that every compact submanifold K of M is contained in a compact set K' of M so that loops in $M - K'$ bound in $M - K$. Since K is compact, it has a finite number of boundary components. Each boundary component F contains a bouquet of a finite number of loops L_F so that $F - L_F$ is an open disk. Since $H_1(M) = 0$, all the loops in all of the L_F bound a finite number of surfaces in M whose union S is compact. Let K' be the compact set $K \cup S$. Let J

be a loop in $M - (K \cup S)$. Since $H_1(M) = 0$, linking numbers for pairs of loops are well defined and symmetric. Since J misses S, the linking number of J and any loop in L_F is zero. Thus, J bounds a surface disjoint from the L_F. If this surface hits K at all, it enters through open disks in Bd K. This surface can then be cut off on Bd K.

PROOF OF THEOREM 2 AND ADDENDUM 2.1. The necessity of the Poincaré conjecture is covered by the example of §3.

We establish some notation. Let $N_i = M - \text{Int } K_i$ and $\hat{N}_i = \hat{M} - \text{Int } K_i$. The $\hat{N}_i$ are the neighborhoods of the nonmanifold set with tori boundary referred to in Theorem 1. Note that hypotheses (ii) and (iii) of Theorem 2 require that each component of these neighborhoods have connected boundary.

We now construct the embedding of M in a compact manifold $\overline{M}$. We first wish to enlarge the "layers" between the successive K_i to obtain various properties. By choosing a subsequence of the K_i and renumbering from 1, the 1-LC property lets us assume that all curves in $\hat{N}_i$ shrink in $\hat{N}_{i-1}$, $i \geqslant 2$. We choose yet a further subsequence.

The boundary of K_3 is compact, so $H_1(\text{Bd } K_3)$ is generated by a finite number of 1-cycles. These bound a finite number of surfaces in N_2 which are contained in some K_{j_1}. Similarly surfaces bounded by generators of $H_1(\text{Bd } K_{j_1+1})$ in N_{j_1} lie in some K_{j_2}. Continuing this way we get a subsequence $K_2, K_3, K_{j_1}, K_{j_1+1}, K_{j_2} \ldots$ which, if renumbered from 1, has the property that (i) loops in N_1 bound in M and (ii) for every even number i, loops in Bd K_i bound surfaces in $K_{i+1} - K_{i-1}$. We will work with this sequence through the remainder of the proof.

The embedding of M in $\overline{M}$ will proceed by successively embedding the layers between the odd numbered K_i. To describe these embeddings, we need certain canonical curves on the components of Bd K_i for all odd $i \geqslant 3$. To keep the notation simple, we will describe these curves for $i = 3$, the process for obtaining them for higher indexes being identical.

Let $Y = \text{Cl}(K_5 - K_1)$. It is most probable that Y is not connected. We show that Y embeds in a 3-manifold Z (not necessarily connected) with $H_1(Z) = 0$. We will use [**21**, Theorem 3.1] which says this can be done if Y is orientable, $H_1(Y)$ is free and $i_*\colon H_1(\text{Bd } Y) \to H_1(Y)$ is onto. Since loops in N_1 bound in M we get immediately that Y is orientable and that $i_*\colon H_1(\text{Bd } Y) \to H_1(Y)$ is onto. From the long exact sequence for $(Y, \text{Bd } Y)$ with reduced homology, $H_1(Y, \text{Bd } Y) \cong \tilde{H}_0(\text{Bd } Y)$ which is free. By Lefschetz duality, $H^2(Y)$ is free, and by the Universal Coefficient Theorem $H_1(Y)$ is free.

We will regard Y as a submanifold of both M and Z. We will refer to Z several times throughout the rest of the proof. Let G be a component of Bd K_3. Since $H_1(Z) = 0$, G separates Z. By hypothesis, K_3 is connected and G separates M, so that if U_G and V_G are the complementary domains of G in Z, one of them, say V_G, does not intersect $K_3 - K_1$. By [**21**, Theorem 2.1], there is a pair of disjoint simple closed curves A_G and B_G on G that intersect in a single point, transversely, and so that B_G is null homologous in $\text{Cl}(V_G)$ and A_G is null homologous in $\text{Cl}(U_G)$. We can find such curves on all components of Bd K_3.

The theorem will follow from:

Claim 1. *Embed* K_{2i+1} *in a compact manifold* $\overline{M}$ *by attaching solid tori to the components* G_j *of* Bd K_{2i+1} *so that the curves* B_{G_j} *are meridians for the solid tori. Then* K_{2i+3} *can be embedded in* $\overline{M}$ *so that* $\mathrm{Cl}(\overline{M} - K_{2i+3})$ *is a union of solid tori whose meridians are the curves* B_G *for components of* Bd K_{2i+3} *and so that the two embeddings agree on* K_{2i-1}.

Again, to keep the notation simple, we will show how an embedding for K_3 implies an embedding of K_5 so that the two embeddings agree on K_1.

First some homology preliminaries. Let G be a component of Bd K_3. G separates Y into two sets, $Q = V_G \cap Y$ and $R = U_G \cap Y$. We know B_G bounds a singular surface in V_G. This surface can be "cut off" on Bd K_4. Since curves on Bd K_4 are null homologous in $K_5 - K_3$, we can find a surface bounded by B_G in Q. Similarly, A_G bounds a surface in U_G. This surface may hit both Bd K_2 and Bd K_4. In the same manner, we get that A_G bounds a surface in R. Note that while we have shown that B_G is null homologous in $K_5 - K_3$, we have not shown that A_G is null homologous in $K_3 - K_1$. We only know that A_G is null homologous in the complementary domain of G in Y that intersects $K_3 - K_1$.

This operation of "cutting off" a surface on an even numbered Bd K_i and redefining the surface in a more restricted area will be done several times during the proof.

Note that as defined above, Q is a component of $K_5 - K_3$. We now state:

Claim 2. *Let* Q *be a component of* $K_{2i+3} - K_{2i+1}$. *Let* $\{G_1, \ldots, G_n\}$ *be the components of* Bd Q *that are in* Bd K_{2i+3}. *Then each* A_{G_i} *is homotopic in* Q *to a product of the curves* $\{B_{G_j}\}$.

We will finish the proof of Claim 1 before giving the proof of Claim 2. Let K_3 be embedded in $\overline{M}$ as given in the statement of Claim 1. Let Q be a component of $\mathrm{Cl}(K_5 - K_3)$. One boundary component F_Q of Q comes from Bd K_3. The other boundary components $\{G_1, \ldots, G_n\}$ of Q come from Bd K_5. Form the space $\overline{Q}$ by sewing solid tori to the $\{G_j\}$ so that meridians are sewn to the curves $\{B_{G_j}\}$. The proof would be finished if $\overline{Q}$ were a solid torus with meridian B_{F_Q}. Such need not be the case however. Note that each component F of Bd K_3 has a component of $K_5 - K_3$ as one of its complementary domains. We shall call its closure Q_F.

Each $\overline{Q}$ can be regarded as a submanifold of $\overline{K}_5$ obtained by sewing solid tori to all components of Bd K_5 so that meridians are sewn to all the B curves. Since every loop in $\hat{N}_3$ bounds a singular disk in $\hat{N}_1$, Claim 2 says that every loop in $\overline{K}_5 - K_3$ bounds a singular disk in $\overline{K}_5 - K_1$. Thus, all the components of Bd K_3 compress to embedded 2-spheres in $\overline{K}_5 - K_1$.

Let $D_1, \ldots, D_m$ be a sequence of embedded disks in $\overline{K}_5 - K_1$ along which all the components of Bd K_3 compress. The compressions are done one at a time, starting with D_1. We can choose each Int D_i disjoint from all components of Bd K_3 as they exist after the first $i - 1$ compressions. Since each component of Bd K_3 is a torus, there are as many disks D_i as components of Bd K_3. We claim that each Bd D_i can be chosen either an A or B curve for the component of Bd K_3 that D_i compresses.

Let F be a component of Bd K_3. If F compresses along a D_i in $\overline{Q_F}$, then by cutting off D_i on Bd K_4 we get Bd D_i zero homologous in Q_F. Thus, Bd D_i is homotopic to B_F on F and F compresses along B_F. If D_i is not contained in $\overline{Q_F}$, it does not hit Q_F and can be cut off on Bd K_2 and Bd K_4 ultimately showing, for reasons of homology, that Bd D_i is homotopic to A_F on F.

Each time a component F is compressed along a disk in $\overline{Q_F}$, a 1-handle is removed from $\overline{Q_F}$ and a 2-handle is added to its complement. Other disks D_i may run through this 2-handle. We can require that the intersection is consistent with the product structure of the 2-handle.

We now switch our attention to the uncompressed components of Bd K_3 as they are embedded in $\overline{M}$. We can compress these components along a sequence of disks $E_1, \ldots, E_m$ that correspond to the disks $D_1, \ldots, D_m$ in the sense that, for each i, $E_i \cap K_3 = D_i \cap K_3$. This can be done, if D_i lies in some $\overline{Q}$, by letting E_i be a meridian of the solid torus bounded, in $\overline{M} - K_3$, by the component of Bd K_3 containing Bd D_i. Otherwise, define $E_i \cap K_3$ to be $D_i \cap K_3$ and fill in the holes of E_i using meridians of solid tori whose boundaries were compressed by earlier disks E_j in the sequence.

Let K_3^* denote K_3 after the compressions along the disks D_i. The space K_3^* is obtained from K_3 by removing 1-handles and adding 2-handles. Let K_3^{**} denote K_3 after the compressions along the disks E_i. The spaces K_3^* and K_3^{**} are naturally homeomorphic. Each of the complementary domains of K_3^{**} in $\overline{M}$ is either a solid torus minus a 1-handle, thus a 3-cell, or a solid torus plus a 2-handle sewn along some A_F. But A_F and B_F intersect transversely in one point for each F. Thus, all the complementary domains are 3-cells. The complementary domains $\overline{Q}^*$ of K_3^* in $\overline{K}_5$ are bounded by one 2-sphere each, and all loops in $\overline{Q}^*$ shrink in $\overline{K}_5$. Thus, each $\overline{Q}^*$ is simply connected and by hypothesis a 3-cell. Claim 1 is completed by mapping the $\overline{Q}^*$ to the complementary domains of K_3^{**} in $\overline{M}$ and reversing the steps of the compression. Since all of the disks D_i and E_i missed K_1, the new embedding agrees with the old on K_1.

Under the hypotheses of Addendum 2.1, we can say that each A_F is a meridian of a solid torus. Also, using an argument of **[23]**, we can say that if an infinite number of Bd K_i compress along A_F in $K_i - K_{i-2}$, i odd, then M is a union of 3-cells. Otherwise, after a certain point, all of the compressing referred to above takes place in the complementary domains $\overline{Q}_i$ which must then be solid tori. Thus, $\overline{M}$ and each $\overline{K}_i$ is the union of two solid tori with transverse meridians and is S^3.

Proof of Claim 2. The following proof works because π_1 and H_1 of a torus are naturally isomorphic. Before we start the proof we mention one item of background. In general, if X and Y are 3-manifolds joined along boundary F, and L is a loop in X which bounds a disk D in $X \cup Y$, then little can be said about the intersections of D and F. In our present setting a great deal can be said. Our first proof was a lengthy sequence of cuts and homotopies that simplified $D \cap F$. What came out were facts about certain subgroups that were true without referring to a fixed D. The ghost of D remains and bits and pieces can be found in what follows. Again, to simplify the notation, we show that the A curves in Bd K_5 are homotopic in $K_5 - K_3$ to products of conjugates of the B curves in Bd K_5.

Referring again to the technique of cutting surfaces off on the even numbered Bd K_i, we point out that if any loop in Bd K_5 bounds a disk in $K_5 - K_3$, it must be a multiple of some A curve, and if any loop in Bd K_5 bounds a disk in $\hat{N}_5$, it must be a multiple of a B curve.

We will need some basic facts about the homological behavior of the A and B curves. Assume a multiple of an A curve on Bd K_5 is zero homologous in $\hat{N}_5$. The surface it bounds can be cut off on Bd K_6 and we would have a multiple of A bounding in $K_7 - K_5$. We know the B curve that is transverse to A bounds, for the same reason, in $K_7 - K_5$. But two curves on the boundary of a 3-manifold with nonzero intersection number cannot both be torsion in H_1 of the 3-manifold. Thus, no multiple of A bounds in $\hat{N}_5$.

We now wish to show that for each component G of Bd K_5, A_G is homologous in $K_5 - K_3$ to combinations of all the B curves except B_G in Bd K_5. We will also show that the B curves in Bd K_5 are independent in $H_1(K_5 - K_3)$. The second fact follows from the first since if there were a dependence relation, the free rank of the image of i_*: $H_1(\text{Bd } K_5) \to H_1(K_5)$ would be less than the genus of Bd K_5. This would violate a standard consequence of duality in 3-manifolds. To show the first statement, we again use the manifold Z with $H_1(Z) = 0$ in which $K_5 - K_1$ embeds. Form a manifold $\bar{Z}$ by removing, for each component G of Bd K_5, the complementary domain V_G of G in Z that misses K_5, and replacing it with a solid torus H_G so that a meridian of H_G is sewn to B_G. According to [**21**, Theorem 2.1], each B_G generates $H_1(U_G)$ where U_G is the complementary domain of G in Z intersecting K_5. Thus, by replacing the domains V_G one by one, we see that $H_1(\bar{Z}) = 0$. If we let $\bar{U}_G$ denote the complementary domain of G in $\bar{Z}$ that intersects K_5, then each A_G is zero homologous in $\bar{U}_G$. If not, some other curve L_G is by [**21**, Theorem 2.1] since $H_1(\bar{Z}) = 0$. The surface bounded by L_G can be adjusted to hit all the solid tori H_{G_i} in meridinal disks. The surface can also be cut off on Bd K_4. Note that the only intersection of this surface with G is L_G. Thus, L_G is homologous in $K_5 - K_3$ to a combination of B curves in Bd K_5 other than B_G. But, each B_{G_j} is zero homologous in V_{G_j} in Z. Thus, each L_G is zero homologous in U_G in Z and must be homotopic to A_G on G.

We now derive some properties of $H_1(\text{Bd } K_5)$ that follow from the fact that $\hat{M}$ is 1-LC. Let H denote $H_1(\text{Bd } K_5)$. There is a natural isomorphism between H and the direct sum of the fundamental groups of the components of Bd K_5. We will define a sequence of subgroups of H, $0 \subset S_1 \subset S_2 \subset \cdots \subset S_n = H$. We let S_1 be the subgroup of H generated by all loops in Bd K_5 that are null homotopic in $\hat{N}_5$. Let S_2 be the subgroup generated by loops in Bd K_5 that are homotopic in $K_5 - K_3$ to products of conjugates of loops in S_1. The group S_3 will be loops homotopic in $\hat{N}_5$ to products of conjugates of loops in S_2. We continue alternating in this fashion. Even numbered groups are loops that are null homotopic in $K_5 - K_3$ mod loops in lower groups and odd numbered groups are defined similarly via $\hat{N}_5$. Since H is finitely generated and since every loop in Bd K_5 bounds a disk in $\hat{N}_3$, there is a finite n such that $S_n = H$. Choose n to be the smallest with this property. For $1 \leqslant i < j \leqslant n$ we cannot have $S_i = S_j$.

We now analyze the structure of the groups S_i. We can represent H as a direct sum $H_A \oplus H_B$ where H_A is the subgroup of H generated by A curves in Bd K_5 and H_B is the subgroup generated by the B curves in Bd K_5. By the homology properties of the A and B curves, S_1 is a subgroup of H_B; call it B_1.

We claim there are sequences of subgroups $B_1 \subset B_3 \subset B_5 \subset \cdots \subset H_B$ and $A_2 \subset A_4 \subset A_6 \subset \cdots \subset H_A$ so that for odd i, $S_i = B_i \oplus A_{i-1}$ and for even i, $S_i = A_i \oplus B_{i-1}$. We further claim that for even i, each loop in A_i is null homotopic in $K_5 - K_3$ mod loops in B_{i-1}. This last statement completes the proof of Claim 2. It will also be true that for odd i, each loop in B_i is null homotopic in $\hat{N}_5$ mod loops in A_{i-1}.

The proof of these claims proceeds by induction. We will show the inductive step for even i. The inductive step for odd i is similar and simpler since each component of $\hat{N}_5$ has connected boundary.

We consider S_{2k}. Let L be a loop in S_{2k}. L lies on some component G of Bd K_5 and can be represented as $aA_G + bB_G$. We claim that bB_G is in B_{2k-1}. Since L is in S_{2k}, there is a genus zero surface C_1 in $K_5 - K_3$, one of whose boundary components is L and which has all other boundary components in $S_{2k-1} = B_{2k-1} \oplus A_{2k-2}$. By the inductive hypotheses, each loop in A_{2k-2} bounds a genus zero surface in $K_5 - K_3$ together with loops in $B_{2k-3} \subset B_{2k-1}$. Thus, C_1 can be replaced by a genus zero surface C_2 in $K_5 - K_3$ with L as one boundary component and loops in B_{2k-1} for all other boundary components. However, aA_G together with multiples of B curves from components of Bd K_5 other than G bounds a 2-chain C_3 in $K_5 - K_3$. Thus, $C_2 - C_3$ is a 2-chain in $K_5 - K_3$ whose boundary lies in H_B. But, the B curves are independent in $H_1(K_5 - K_3)$, so the coefficient of each B in the boundary of $C_2 - C_3$ is zero. But $\mathrm{Bd}(C_2 - C_3) \cap G$ is $(\mathrm{Bd}\ C_2 \cap G) - aA_G$ since C_3 hits G only in aA_G. The boundary of C_2 on G is $aA_G + bB_G$ + elements in B_{2k-1}. So $\mathrm{Bd}(C_2 - C_3) \cap G$ is bB_G + elements in B_{2k-1}. Thus, bB_G is in B_{2k-1}. Now the genus zero surface C_2 shows that aA_G is null homotopic in $K_5 - K_3$ mod loops in B_{2k-1} and the proof of Claim 2 is complete.

3. If the Poincaré conjecture is false. We start with a description of the example. Let P be a prime homotopy 3-sphere that is not homeomorphic to S^3. Let C be a 3-cell in P containing a spanning arc A and a simple, closed curve J_1 arranged as in Figure 1. J_1 bounds an embedded disk in C but not in $C - A$.

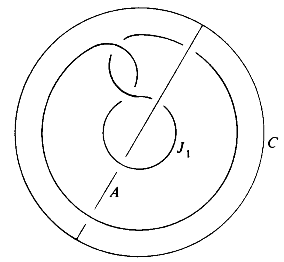

FIGURE 1

Let J_2 be a simple closed curve in P contained in no 3-cell (see [3]) so that $J_2 \cap C = A$. Let T_i be a solid torus regular neighborhood of J_i, $i = 1, 2$. Let $F = \text{Bd } T_1$ and $G = \text{Bd } T_2$.

Let $Q = \text{Cl}(P - T_1 - T_2)$. The boundary of Q is $F \cup G$.

We have $\pi_1(P - J_1) = \pi_1(P - C) * \pi_1(C - J_1) = \{1\} * Z = Z$. Since $\text{Link}(J_1, J_2) = 0$, the curve J_2 must be homotopically trivial in $(P - J_1)$. There is, therefore, a simple closed curve Y in G that bounds a surface in Q. Y is parallel to J_2 in T_2 and transverse to a meridian Z of T_2 (see Figure 2).

The curve J_1 bounds a surface in $P - T_2$, so there is a curve X in F, parallel to J_1 in T_1, that bounds a surface in Q. X is transverse to a meridian W of T_1. The curve X and two parallel copies of Z bound an embedded disk with two holes in $C - \text{Int } T_2$ (see Figure 2).

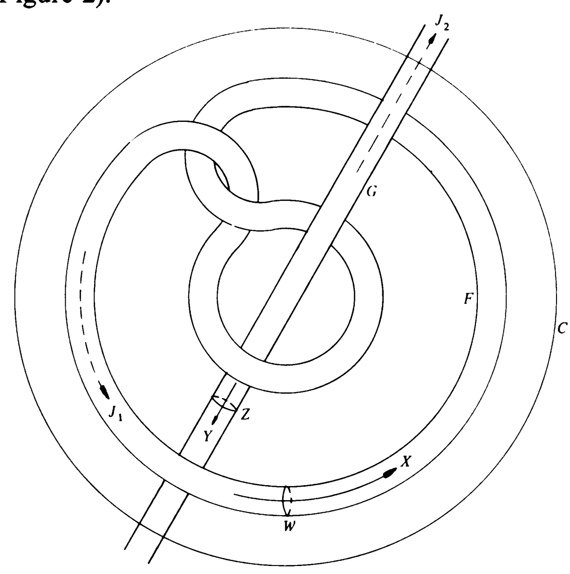

FIGURE 2

Let $Q_1, Q_2, \ldots$ be disjoint spaces homeomorphic to Q with W_i, X_i, Y_i, Z_i, F_i and G_i in Q_i corresponding to W, X, Y, Z, F and G in Q respectively. Let $M = T \cup Q_1 \cup Q_2 \cup \ldots$ with Q_i sewn to Q_{i+1} by identifying (G_i, Y_i, Z_i) to $(F_{i+1}, W_{i+1}, X_{i+1})$, and with T a solid torus whose boundary is sewn to F_1 so that a meridian is sewn to W_1.

Let $\hat{M}$ be the 1-point compactification of M.

THEOREM 3. *The space $\hat{M}$ is a compact generalized 3-manifold such that*

(i) *the nonmanifold set consists of one point with neighborhoods bounded by tori,*

(ii) *the manifold set is irreducible, and*

(iii) *the manifold set embeds in no compact 3-manifold.*

PROOF. First some notation. Let p be the point in $\hat{M} - M$. We will use K_n to denote $T \cup Q_1 \cup \cdots \cup Q_{n-1}$. We will use N_n to denote $\text{Cl}(M - K_n)$ and $\hat{N}_n$ to denote $\text{Cl}(\hat{M} - K_n)$. Note that Q_n is contained in N_n and $\hat{N}_n$ but not in K_n. The proof is divided into a number of steps. The first few steps, and the bulk of the proof, concern themselves with the local contractibility of $\hat{M}$. The remaining steps cover the local homology properties. The primary reasons that $\hat{M}$ is locally contractible seem to be that $P - J_1$ has trivial π_2 and that each Bd $\hat{N}_i$ bounds a finite 3-chain in $\hat{N}_i$. These combine to give the local 2-connectedness, the major stumbling block.

Step 1, a preliminary, proves item (ii). After that our attention focuses on the point p, the only point in the nonmanifold set.

Step 1. *The manifold set is irreducible*. We show that G is incompressible in $\mathrm{Cl}(P - T_2)$ and F is incompressible in Q. If G compresses, it must compress along Y, implying that J_2 is contained in a 3-cell, contradicting our choice of J_2. If F compresses, it must compress along X, implying that J_1 bounds an embedded disk D in $P - J_2$. If L is a simple closed curve of Bd $C \cap D$, then L does not link J_2. Thus, one of the disks bounded by L in Bd C misses J_2, and D can be cut off on Bd C. This implies that J_1 bounds an embedded disk in $C - A$, contradicting our choice of A and J_1 in C.

Since J_2 is contained in no 3-cell and since P is prime, $T \cup Q_1$ and all Q_i, $i \geqslant 2$, are irreducible. Also, $T \cup Q_1$ and all Q_i, $i \geqslant 2$, have incompressible boundary. Thus, every K_i is irreducible and M is irreducible.

Step 2. *$\hat{M}$ is* 1-*LC at P*. We observe first that every Z_i bounds a disk in $\hat{N}_{i+1}$ since Z_i is X_{i+1}, and X_{i+1} and two copies of Z_{i+1} bound a disk with two holes in Q_{i+1}. The holes can be partly filled in by disks with two holes in Q_{i+2}. Continuing in this way we fill in the disk by mapping a Cantor set to p.

Also note that W_i (which equals Y_{i-1}) bounds a disk in $\hat{N}_{i-1}$. Temporarily regarding Q_{i-1} as a subset of P and recalling that J_2 was homotopic to zero in $P - J_1$, we see that Y_{i-1} with several copies of Z_{i-1} bounds a disk with holes in Q_{i-1}. These holes can be filled in with disks in $\hat{N}_i$ as above.

Now any loop in Q_i will bound a disk with holes together with copies of W_i and Z_i. Thus, any loop in Q_i bounds a disk in $\hat{N}_{i-1}$. What follows is essentially an argument of Kozlowski.

Let L be a loop in $\hat{N}_i$. By a small homotopy, put L in general position with respect to all the Bd N_j. By the previous paragraphs, there is a homotopy, contained in $\hat{N}_{i-1}$, taking each arc of $L \cap Q_i$ to an arc in Bd N_{i+1} keeping the endpoints fixed. Now L lies in $\hat{N}_{i+1}$. Similarly a homotopy contained in $\hat{N}_i$ carries L into $\hat{N}_{i+2}$. Continuing, we can put these homotopies together to define a homotopy on $[0, 1)$. Since the homotopies near 1 can be kept arbitrarily close to p, we can extend the homotopy continuously to all of $[0, 1]$ by making the 1 level the constant map to p.

Step 3. *For each* $i \geqslant 1$, *there is a map* $f_i\colon T \to \hat{N}_i$, *where* T *is a solid torus and* $f_i|_{\mathrm{Bd}\, T}\colon \mathrm{Bd}\, T \to \mathrm{Bd}\, \hat{N}_i$ *is a homeomorphism*.

This will be needed to show that $\hat{M}$ is 2-LC at p. Since all the $\hat{N}_i$ are homeomorphic, we will construct the map f_1.

Divide P into the 3-cell C and its complement C'. The space $\mathrm{Cl}(C - T_1)$ is homeomorphic to the solid torus T minus the interior of a 3-cell B in Int T. There is a map from the 3-cell B to C' taking the boundary homeomorphically to the boundary. Putting these together gives a map f from T to $\mathrm{Cl}(P - T_1)$ that takes the boundary homeomorphically to the boundary. We can put f in general position with respect to J_2.

The set $f^{-1}(J_2)$ consists of a finite number of disjoint simple closed curves $L_1, \ldots, L_n$ with all but one, say L_1, contained in B. The curve L_1 intersects B in a spanning arc, possible knotted. If $L_1 \cap B$ is replaced by an unknotted spanning arc, creating a curve L_1^*, then the embedding of L_1^* in T is that of the standard Whitehead link (see Figure 3).

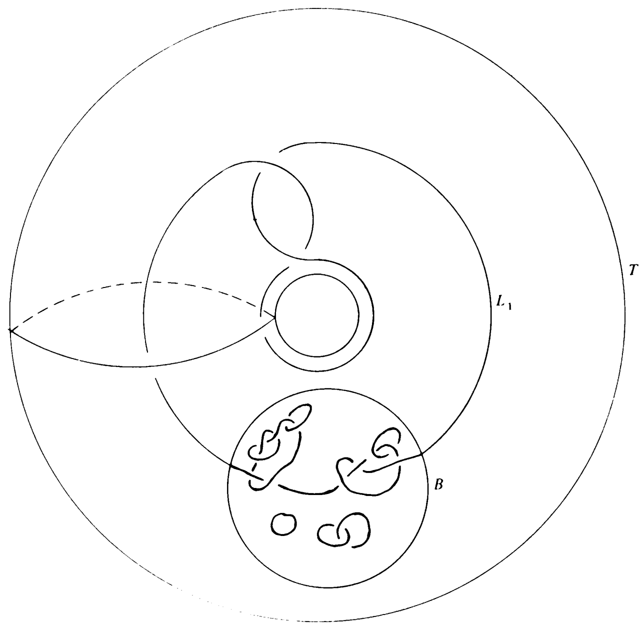

FIGURE 3

We can adjust f by a homotopy on a small neighborhood of $f^{-1}(J_2)$ so that a neighborhood T_L of L_1 is carried homeomorphically onto T_2 and so that $f^{-1}(T_2)$ is a regular neighborhood of $f^{-1}(J_2)$ one of whose components is T_L. The space $T - f^{-1}(T_2)$ is carried by f onto Q_1. We wish to redefine f on $f^{-1}(T_2)$ so the new map takes $f^{-1}(T_2)$ to $\hat{N}_2$. We will do this first on neighborhoods of L_i for $i > 1$.

Let T_i be the component of $f^{-1}(T_2)$ that contains L_i. Note that f carries Bd T_i into an annulus of G_1, the boundary component that Q_1 shares with Q_2. This annulus is a neighborhood of Z_1 on G_1. We can adjust f by a homotopy so that f carries Bd T_i into Z_1. Since Z_1 bounds a disk in $\hat{N}_2$, we can extend the map to take T_i to $\hat{N}_2$ by first extending the map to map T_i into a 2-disk bounded by Z_i by the Tietze extension theorem and then mapping the disk into $\hat{N}_2$ with boundary Z_1.

This leaves the solid torus neighborhood of L_1. We can fill that in by mapping that solid torus minus a "knotted Whitehead link" into $\hat{N}_2$ in a manner identical to the above. Continuing this way gives a map defined on all of T except on a certain cell-like continuum. Points close to that continuum are mapped close to p. The map f_1 is completed by mapping the continuum to p. Note that $f^{-1}(p)$ contains components other than this continuum, namely, simple curves essential in solid tori that are contained in 3-cells in T.

Step 4. $\hat{M}$ *is* 2-*LC at* p. Let f: $S^2 \to \hat{N}_i$ be a singular map of a 2-sphere into $\hat{N}_i$. We wish to show that f extends to a map of I^3 into $\hat{N}_{i-1}$ where S^2 is Bd I^3. We will do this in pieces.

Put f in general position with respect to the surfaces Bd N_j. Each simple closed curve L of $f^{-1}(\text{Bd } N_j)$, all j, bounds an embedded disk D_L in I^3 with $D_L \cap S^2 = L$. Since all the curves L are disjoint, we can choose these disks disjoint (there are a countable number). Fix an L in $f^{-1}(\text{Bd } N_j)$. We wish to extend f to D_L very explicitly. Since Bd N_j is a torus with π_1 generated by W_j and X_j, $f(L)$ is homotopic to a loop L' that follows W_j around n times and then X_j around m times, with x, the point $W_j \cap X_j$, used as a base point. We can homotop f near L so that f takes L to such a loop. Choose two points x_1 and x_2 in $f^{-1}(x)$ so that one complementary domain of x_1 and x_2 in L goes to nW_j and the other to mX_j.

Including x_2, there are $m + n - 1$ points in $f^{-1}(x)$ other than x_1. Join these $m + n - 1$ points to x_1 by arcs in D_L that are disjoint except at x_1 and whose interiors lie in Int D_L. Extend the map f to these arcs by defining it to be the constant map to x on them. The disk D_L has been carved into $m + n$ subdisks. Each of their boundaries maps either to W_j or to X_j. We extend f to these subdisks of D_L exactly as specified in the first two paragraphs of Step 2.

We will now regard f as being defined on S^2 and all the disks D_L. The disks D_L carve I^3 into a countable number of 3-cells with f defined on the boundary of each. Note that f carries each D_L only one neighborhood farther from p than it carries Bd D_L. Thus, for all j, $f^{-1}(\hat{N}_j)$ contains the boundaries of all but a finite number of these 3-cells. We will be done if we can extend f to each of these 3-cells so that f does not carry the interior much farther from p than it carries its boundary.

We now, therefore, reduce the problem and regard I^3 as one of the subcells whose boundary consists of a connected surface S_0' in the original 2-sphere plus a collection of disks D_L. The map f takes S_0' into some Q_i with the interior of S_0' carried into the interior of Q_i. The boundaries of the subdisks of each D_L are carried to the curves W_i, X_i, Y_i and Z_i. Recall that the disks constructed in Step 2 bounded by X_i and Y_i had neighborhoods of their boundaries mapped into Q_i. Thus, if we move f by a small homotopy that pulls the images of the arcs in D_L defined above into the interior of Q_i, we get a different description of Bd I^3. We now have a genus zero surface S_0 in Bd I^3 obtained from S_0' by adding neighborhoods of those arcs in the disks D_L that were part of the boundaries of subdisks whose boundaries went to W_i or Z_i under f. Also, added to S_0' are those parts in Q_i of the subdisks of D_L whose boundaries went to X_i and Y_i under f.

We now have S_0 mapped to Q_i by f, $f(\text{Bd } S_0) \subset \text{Bd } Q_i$, $f(\text{Int } S_0) \subset \text{Int } Q_i$, and each boundary component of S_0 mapped to curves parallel to W_i or Z_i. The disks of (Bd I^3) $-$ S_0 are the disks described in Step 2 bounded by curves parallel to W_i and Z_i. Note that portions of Bd I^3 other than S_0 are mapped into Q_i, namely parts of the disks constructed in Step 2 bounded by curves parallel to W_i.

We would like to extend $f|_{\mathrm{Bd}\, I^3}$ to all of I^3. We will do that by first constructing a map which takes a 3-cell into $\hat{N}_i$ and which takes the boundary of the 3-cell to a set somewhat different from $f(\mathrm{Bd}\, I^3)$. We will then show how to use this new map to extend f to all of I^3.

Orient Z_i, W_i and S_0. Orient the boundary components of S_0 consistently with S_0. Since Z_i and W_i are independent generators of $H_1(Q_i) = Z \oplus Z$, there must be as many boundary components of S_0 mapping to curves on G_i parallel to Z_i as antiparallel to Z_i. Similarly, for W_i on F_i. Thus, we can add annuli between pairs of components of Bd S_0 that map to G_i and send the annuli into G_i, and we can add annuli between pairs of components of Bd S_0 that map to F_i and send these annuli into F_i to create a closed orientable singular surface S_1 that maps into Q_i.

We can regard Q_i as a submanifold of P with S_1 mapping into Bd T_1 and Bd T_2 along the annuli added in the last paragraph. We can remove the annuli mapping into Bd T_2 and cap off the holes with disks that map onto meridinal disks of T_2. This only partially reconstructs a sphere, leaving us with a surface that we will call S_2. We have to be more careful in dealing with T_1.

Let A be an annulus added in the creation of S_1 that maps into Bd T_1. Let E be an arc in A joining its two boundary components. Let E' be an arc in S_0 connecting the endpoints of E. We can do this for all the annuli mapping into Bd T_1 keeping the arcs E' disjoint since S_0 is a sphere with holes.

Let $E \cup E'$ be one of the simple closed curves on S_2 just constructed. Its image may not shrink in $P - \mathrm{Int}\, T_1$. However, W generates $\pi_1(P - \mathrm{Int}\, T_1)$ and the annulus containing E contains a curve W' that maps to a curve parallel to W and that pierces E once. Thus, E can be replaced by another arc E'' in the annulus by adding multiples of W', so that the image of $E'' \cup E'$ bounds a singular disk in $P - \mathrm{Int}\, T_1$. Do this for each annulus.

We will now compress the surface S_2 by replacing annular neighborhoods of the curves $E'' \cup E'$ by pairs of disks that map into $P - \mathrm{Int}\, T_1$. This gives us a sphere S_3 that maps into $P - \mathrm{Int}\, T_1$. We can make sure that the images of the new disks hit T_2 in meridional disks.

Since P is a homotopy 3-sphere and T_1 is connected, the sphere theorem **[17]** says that $\pi_2(P - \mathrm{Int}\, T_1) = 0$. Thus, S_3 bounds a 3-cell that maps into $P - \mathrm{Int}\, T_1$. We have already identified Q_i with $\mathrm{Cl}(P - T_1 - T_2)$. We can map T_2 to $\hat{N}_{i+1}$ by Step 3. Recall that this map was badly behaved only on certain 3-cells. We can insure that these 3-cells do not intersect the meridional disks in T_2 that are in the image of S_3.

Certain meridional disks of Image$(S_3) \cap T_2$ were meridional disks of Image$(S_2) \cap T_2$. These will be carried onto the disks in $\hat{N}_{i+1}$ capping off those components of Bd S_0 mapped to curves parallel to Z_i.

We have carried S_3 and the 3-cell it bounds into $\hat{N}_i$. We wish to show that $f(\mathrm{Bd}\, I^3)$ bounds a 3-cell. Unfortunately, S_3 and Bd I^3 do not coincide. We will analyze their differences. In what follows, certain letters will represent collections of objects.

To obtain S_3 from Bd I^3, pairs of disks D whose boundaries map to curves parallel to W_i are removed and replaced by annuli A. Annuli $\tilde{A}$ that are neighborhoods of the curves $E'' \cup E'$ are removed and replaced by pairs of disks $\tilde{D}$. The annuli A and the annuli $\tilde{A}$ are paired, one from A with one from $\tilde{A}$, with each associated pair intersecting transversely in a disk and all other combinations disjoint.

For each pair in $\tilde{D}$, the images of the two disks are parallel off p since the map from T_2 to $\hat{N}_{i+1}$ is well behaved off certain 3-cells, and the parallel pairs can be kept close together in T_2 so that the 3-cells containing the bad part of the map of Step 3 do not get between them.

Thus, each annulus from $\tilde{A}$ with its associated pair of disks from $\tilde{D}$ bounds a 3-cell mapping into $\hat{N}_i$ which can be added to the 3-cell bounded by S_3 as a 1-handle. This removes the disks $\tilde{D}$ from S_3 and adds in the annuli $\tilde{A}$. This turns the 3-cell bounded by S_3 into a handlebody mapping into $\hat{N}_i$.

We now point out that the disk pairs D map to parallel copies of the disks of Step 2 bounded by W_i. Each of these pairs together with the appropriate annulus from A bounds a singular 3-cell mapping into $\hat{N}_{i-1}$. Each of these 3-cells can be added to the handlebody of the last paragraph as a 2-handle. By this process Bd I^3 is recovered. The 1-handles and 2-handles are transverse in pairs and cancel. Thus, a 3-cell that maps into $\hat{N}_{i-1}$ and is bounded by Bd I^3 is created. There is a map from I^3 to this 3-cell that is the identity on Bd I^3. Step 4 is complete.

Step 5. *$\hat{M}$ is locally contractible.* Each $\hat{N}_i$ collapses to a subset of its 2-skeleton $\cup \{p\}$. We thank D. R. McMillan for pointing this out. One way to accomplish this is to take a triangulation of N_i small with respect to the various Q_j and start collapsing at Bd N_i. We can demand that all 3-simplexes in a Q_j be collapsed before starting on any in Q_{j+2}. Thus, points near p will not be moved far from p in the process. This much care is not really needed since each Q_j is compact and there are an infinite number of $\hat{N}_k$.

It is now a standard exercise to pull the various skeleta near p into p within a slightly larger neighborhood.

We now look at the local homology properties of $\hat{M}$. By excision it is sufficient to show $H_*(\hat{N}_i, N_i) \cong H_*(E^3, E^3 - \{0\})$ for one i. We will consider the following exact sequence:

$$H_3(\hat{N}_i) \to H_3(\hat{N}_i, N_i) \to H_2(N_i) \to H_2(\hat{N}_i) \to H_2(\hat{N}_i, N_i)$$
$$\to H_1(N_i) \xrightarrow{i} H_1(\hat{N}_i) \to H_1(\hat{N}_i, N_i) \to 0.$$

We will use the following observation. Since each $\hat{N}_j$ contracts in some $\hat{N}_{j-k}$ (k fixed), then each n-cycle in $\hat{N}_j$ is homologous to an n-cycle in Bd $\hat{N}_j$. This is geometrically obvious and can be shown using excision.

Step 6. *The groups $H_3(\hat{N}_i)$, $H_2(\hat{N}_i)$ and $H_1(M)$ are trivial and $H_2(N_i) \cong Z$.* The first two we get from the above observation, from the fact that $H_3(\text{Bd } \hat{N}_i) = 0$ and from the map of Step 3. We get $H_1(M) = 0$ since every loop in Q_i bounds mod W_i and Z_i which each bound in Q_{i-1} and Q_{i+1} respectively. Since $H_1(M) = 0$, every surface in M separates. Since M has one end, every surface bounds. We can show $H_2(N_i) \cong Z$ by the Mayer-Vietoris sequence.

Step 7. *The map i_* is an isomorphism.* Local simple connectivity shows that i_* is surjective. Loops passing through p can be pulled slightly off p.

To show i_* is injective we consider a loop L that bounds a surface in $\hat{N}_i$. By Lemma 2.2, M is 1-acyclic at infinity. Thus, there is an N_j so that every loop in N_j bounds in N_i. Since L is compact, N_j can also be chosen to miss L. The surface bounded by L can be cut off on Bd N_j and replaced by a surface completely contained in N_i.

Step 8. *The space $\hat{M}$ is a generalized* 3-*manifold.* The local homology properties follow from the information filled in the long exact sequence. Also a finite dimensional, locally contractible, separable metric space is an ANR and in fact a retract of an open subset of a finite dimensional Euclidean space (see [**4**, p. 122]).

Step 9. *The* 3-*manifold M embeds in no compact* 3-*manifold.* We use a result of [**23**] that is based on Haken's finiteness theorem for incompressible surfaces [**9**, §4]. Since each Q_i has incompressible boundary, [**23**] says that if M embeds in a compact 3-manifold X, then all but a finite number of $\mathrm{Cl}(X - K_i)$ would be solid tori. Since $X - M$ would be cell-like (see introduction), the map from a solid torus $\mathrm{Cl}(X - K_i)$ to $\hat{N}_i$ gotten by mapping $X - M$ to p would be an isomorphism on π_1 [**12**]. Thus, a meridian for $\mathrm{Cl}(X - K_i)$ would go to a curve on Bd $\hat{N}_i$ trivial in $\hat{N}_i$. Such a curve must be X_i. Since all but a finite number of the $\mathrm{Cl}(X - K_i)$ are solid tori, we can find an i and $i + 1$ where this holds.

Thus, $\mathrm{Cl}(X - K_{i+1})$ is a solid torus with meridian X_{i+1}. But X_{i+1} is Z_i, and Q_i plus a solid torus with meridian sewn to Z_i is $P - T_1$. This would say that $P - T_1$ is homeomorphic to $\mathrm{Cl}(X - K_i)$, a solid torus, with X_i as a meridian. But this would imply $P \cong S^3$, contradicting our choice of P.

This proves Theorem 3.

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